Symmetry Analysis of Some Fractional Order Partial Differential Equations

Synopsis

of

Ph.D. thesis

submitted by

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Preface

During the past few decades fractional calculus has grown predominantly in pure mathematics as well as in scientific applications. But to classify fractional calculus as a young science would be completely wrong. The concept of fractional derivative is almost as old as its integer order counterpart and was originated in a letter correspondence between Leibniz and L. Hospital in 1695, where the two mathematicians discussed about the derivative of order one half [1,2]. After that many mathematicians have worked in this old yet novel field of mathematical science. Several theoretical applications of the fractional derivatives have been found during their long history. In many applications, it is presumed that the future state of a system is independent of the past state and determined entirely on the present. But now it has been recognised that this assumption leads to first approximation of the true situations [3,4]. Therefore, for a better approximation one has to consider also the past history of the system. This can be achieved by using the fractional order differential operator which is not local in nature, i.e., the derivative depends on the whole history of the function. Many processes in physics and engineering can be modeled more accurately by fractional derivatives or fractional integrals than the traditional integer order derivatives or integrals [5-7]. Miller and Ross specified that fractional derivatives have applications in almost every field of science and engineering. For example, one way to develop physically meaningful models for anomalous diffusion is to derive the limiting distribution of an ensemble of particles following a specified stochastic process. The continuous time random walks where each random particle jump occurs after a random waiting time can be used to derive these limits. Very large particle jumps are associated with fractional derivatives in space, while very long waiting times lead to fractional derivatives in time. For such models, the limiting particle distribution is governed by a fractional differential equation involving space and time fractional derivative operators [8-10]. Many important phenomena in viscoelasticity, electromagnetics, material science, acoustics and electrochemistry have been elegantly described with the help of fractional order differential equations [11-13]. The exact solutions of these fractional differential equations play a central role in the theories of these physical phenomena and have become more and more sought after during last few decades. Lie group method is one of the mathematical techniques which is applicable to all types of differential equations to furnish a variety of exact solutions in a systematic manner [14,15]. The investigations carried out in this work are confined to the applications of Lie group method to five nonlinear partial differential equations of fractional order.

Objectives

- To study the symmetry properties and reduction to ordinary differential equations of some fractional order partial differential equations.
- To obtain the group invariant and other exact solutions of these systems of fractional differential equations.

Present Work

The thesis comprises seven chapters. The brief outline of each chapter is as under:

Chapter 1: Introduction

The study of differential equations has been playing a central role in the development of mathematics and its applications in almost every branch of science and engineering for nearly five centuries. Most of the problems posed by nature are characteristically nonlinear and are often represented by a single or a system of partial order differential equations. Since then the integer order partial differential equations have been a powerful tool in order to model and study the dynamics of many physical processes of the applied sciences. But nature often presents complex dynamics, which cannot be explained by means of ordinary models and from the experimental observations and reality, it has been revealed that there exists a lot of complex systems in nature which have anomalous dynamics such as the transport of chemical contaminants, the dynamics of viscoelastic materials as polymers, network traffic, financial markets and many more. In most of the above mentioned cases, their dynamics cannot be characterized by classical derivative models. During the last four decades the fractional derivatives have been proved to be valuable tools in modelling such physical phenomena. When compared with variety of methods available to solve a system of integer order partial differential equations, the tools for analysis of fractional order partial differential equations are limited to some very special categories (see for example [16-28]). In a sense fractional order systems must be treated in toto and in their full complexity, and so it is not surprising that there exists no general method for solving them. Exact solutions for fractional order differential equations are rare, and the methods, which can generate families of them, are not only increasingly popular, but more and more sought.

The prime objective and motivation behind the proposed study is to demonstrate the importance and efficacy of symmetry group methods in solving fractional systems. The method is based on the theory of continuous group of point transformations acting on the space of independent and dependent variables of the system and was introduced originally by Sophus Lie(1882). Lie established that the order of an ordinary differential equation can be reduced by one if it is invariant under a one parameter symmetry group and for a partial differential equation the invariance under a continuous group of transformations leads directly to superposition of solutions in terms of transformations. Further, Olver [15], Bluman and Cole [29] and Ovsiannikov [30] extended the theory of Lie groups to wide range of problems. In brief, a symmetry group of a single or a system of partial differential equation of fractional order is a continuous group of transformations acting on the space of independent and dependent variables which leaves the equation(s) invariant. This group can be determined algorithmically and then the solutions of fractional order partial differential equation(s) can be found by solving a reduced system of ordinary differential equations of fractional order.

We would like to mention that there is no unique definition to define the fractional derivative. In the literature, different definitions for the fractional derivative such as the Caputo, the Riesz, the Grunwald-Letnikov and the Riemann-Liouville can be seen. The most popular ones are the Riemann-Liouville and the Caputo derivatives. Each fractional derivative presents some advantages and disadvantages (refer to [3,4,12]). Recently, Guy Jumarie [31-33] proposed a simple alternative definition to the Riemann-Liouville derivative. His modified Riemann-Liouville derivative has the advantages of both the standard Riemann-Liouville and Caputo fractional derivatives: it is defined for arbitrary continuous (non-differentiable) functions.

Chapter 2: Group invariant solutions of fractional order Burgers-Poisson equation[1]

The application of Lie symmetries is one of the most effective techniques in solving nonlinear partial differential equations (PDEs). Only few researchers have applied the Lie group method on fractional differential equations. In 2010, the fractional Lie group method and the fractional characteristic method are proposed by Wu to solve anomalous diffusion equations [27]. In this chapter, we present the application of fractional Lie group method to a fractional order Burgers-Poisson equation(FBP equation).

[1] The work is published as "On group invariant solutions of fractional order Burgers-Poisson equation," *Applied Mathematics and Computation*, 244.1, pp. 870-877, 2014.

We considered the equation

$$u_t^{(\alpha)} - (u_x^{(2\beta)})_t^{\alpha} + u_x^{(\beta)} + uu_x^{(\beta)} - 3(u_x^{(\beta)}u_x^{(2\beta)} + uu_x^{(3\beta)}) = 0, \tag{2.1}$$

where $x \in (0, \infty)$, $t > 0$, $0 < \alpha, \beta < 1$.

We assume the Lie symmetries of the form

$$\frac{\tilde{x^{\beta}}}{\Gamma(1+\beta)} = \frac{x^{\beta}}{\Gamma(1+\beta)} + \epsilon\xi(x,t,u) + o(\epsilon^2) \tag{2.2}$$

$$\frac{\tilde{t^{\alpha}}}{\Gamma(1+\alpha)} = \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \epsilon\tau(x,t,u) + o(\epsilon^2) \tag{2.3}$$

$$\tilde{u} = u + \epsilon\eta(x,t,u) + o(\epsilon^2), \tag{2.4}$$

where ϵ is the group parameter and ξ, τ and η are the infinitesimals of the transformations for the independent and dependent variables, respectively. The associated Lie algebra and its fractional third order prolongation is

$$V = \xi(x,t,u)\frac{\partial^{\beta}}{\partial x^{\beta}} + \tau(x,t,u)\frac{\partial^{\alpha}}{\partial t^{\alpha}} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{2.5}$$

$$pr^{(3)}V = \xi(x,t,u)\frac{\partial^{\beta}}{\partial x^{\beta}} + \tau(x,t,u)\frac{\partial^{\alpha}}{\partial t^{\alpha}} + \eta(x,t,u)\frac{\partial}{\partial u} + \eta^{t}\frac{\partial}{\partial u_t^{(\alpha)}} + \eta^{x}\frac{\partial}{\partial u_x^{(\beta)}} + \eta^{xx}\frac{\partial}{\partial u_x^{(2\beta)}} + \eta^{xxt}\frac{\partial}{\partial (u_t^{(\alpha)})_x^{(2\beta)}} + \eta^{xxx}\frac{\partial}{\partial u_x^{(3\beta)}} \tag{2.6}$$

Now for the invariance of eqn. (2.1) under equations (2.2)-(2.4), we must have

$$pr^{(3)}V([\Delta u])|_{[\Delta u]=0} = 0, \tag{2.7}$$

where $[\Delta u] = u_t^{(\alpha)} - (u_x^{(2\beta)})_t^{\alpha} + u_x^{(\beta)} + uu_x^{(\beta)} - 3(u_x^{(\beta)}u_x^{(2\beta)} + uu_x^{(3\beta)})$,
or, equivalently

$$(\eta(u_x^{(\beta)} - u_x^{(3\beta)}) + \eta^{x}(1 + u - 3u_x^{(2\beta)}) + \eta^{t} - 3u_x^{(\beta)}\eta^{xx} - u\eta^{xxx} - \eta^{xxt})|_{([\Delta u])=0} = 0. \tag{2.8}$$

Using the generalised fractional prolongation vector fields in eqn. (2.8) and equating the coefficient of various derivative terms to zero, we get the following simplified set of determining equations

$$\tau_u = 0 \tag{2.9}$$

$$\tau_x^{(\beta)} = 0 \tag{2.10}$$

$$\xi_u = 0 \tag{2.11}$$

$$\eta_{uu} = 0 \tag{2.12}$$

$$\eta_u + \tau_t^{(\alpha)} = \xi_x^{(\beta)} \tag{2.13}$$

$$\tau_t^{(\alpha)} = \xi_x^{(\beta)} \tag{2.14}$$

$$\xi_x^{(2\beta)} - (\eta_u)_x^{(\beta)} = 0 \tag{2.15}$$

$$2(\xi_x^{(\beta)})_t^{(\alpha)} - (\eta_u)_t^{(\alpha)} - 3\eta_x^{(\beta)} = 0 \tag{2.16}$$

$$\xi_x^{(3\beta)} - 2(\eta_u)_x^{(2\beta)} - \eta_x^{(2\beta)} + \xi_x^{(\beta)} + \tau_t^{(\alpha)} = 0 \tag{2.17}$$

$$\eta_t^{(\alpha)} - f(t)\eta_x^{(2\beta)} = 0 \tag{2.18}$$

On solving these equations we obtain the infinitesimals

$$\xi = c\frac{t^\alpha}{\Gamma(1+\alpha)} + b, \tag{2.19}$$

$$\tau = a, \tag{2.20}$$

$$\eta = c, \tag{2.21}$$

where a, b, c are arbitrary constants.
Hence, the fractional point symmetry generators admitted by the eqn. (2.1) are given by

$$V_1 = \frac{\partial^\alpha}{\partial t^\alpha}$$

$$V_2 = \frac{\partial^\beta}{\partial x^\beta}$$

$$V_3 = \frac{t^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\beta}{\partial x^\beta} + \frac{\partial}{\partial u}$$

It follows that the nonlinear FBP equation admits a three parameter symmetry group. It may be noted that for $\alpha = \beta = 1$, the infinitesimals reported for the integer order model can be recovered. It can be verified easily that the set $\{V_1, V_2, V_3\}$forms a three dimensional Lie algebra under the Lie bracket $[X, Y] = XY - YX$. Lie group of local point transformations generated by the vector field $V_i, i = 1, 2, 3$ and V, where $V = rV_1 + V_3$ is obtained by solving the system of ordinary differential equations

$$\frac{(d\tilde{x})^\beta}{\Gamma(1+\beta)d\epsilon} = \xi(\tilde{x}, \tilde{t}, \tilde{u}) \tag{2.22}$$

$$\frac{(d\tilde{t})^\alpha}{\Gamma(1+\alpha)d\epsilon} = \tau(\tilde{x}, \tilde{t}, \tilde{u}) \tag{2.23}$$

$$\frac{d\tilde{u}}{d\epsilon} = \eta(\tilde{x}, \tilde{t}, \tilde{u}), \tag{2.24}$$

with the initial conditions

$$\tilde{x}|_{\epsilon=0} = x \tag{2.25}$$

$$\tilde{t}|_{\epsilon=0} = t \tag{2.26}$$

$$\tilde{u}|_{\epsilon=0} = u \tag{2.27}$$

On solving the above equations, we get the one parameter groups

$$g_1 : \left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)}, u\right) \to \left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)} + \epsilon, u\right)$$

$$g_2 : \left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)}, u\right) \to \left(\frac{x^\beta}{\Gamma(1+\beta)} + \epsilon, \frac{t^\alpha}{\Gamma(1+\alpha)}, u\right)$$

$$g_3 : \left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)}, u\right) \to \left(\frac{x^\beta}{\Gamma(1+\beta)} + \epsilon\frac{t^\alpha}{\Gamma(1+\alpha)}, \frac{t^\alpha}{\Gamma(1+\alpha)}, u + \epsilon\right)$$

$$g : \left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)}, u\right) \to \left(\frac{x^\beta}{\Gamma(1+\beta)} + \epsilon\frac{t^\alpha}{\Gamma(1+\alpha)}, \frac{t^\alpha}{\Gamma(1+\alpha)} + r\epsilon, u + \epsilon\right)$$

we also investigated some exact solutions of FBP equation or reduced it to an ordinary differential equation corresponding to the following infinitesimal generators
(i)V_1
(ii)V_2
(iii)V_3
(iv)$rV_1 + V_3$,
where r is an arbitrary nonzero parameter.
Consider the infinitesimal generator V_3, given by

$$V_3 = \frac{t^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\beta}{\partial x^\beta} + \frac{\partial}{\partial u}$$

We find the resulting invariant solution by reducing eqn. (2.1) to a linear ordinary differential equation using differential invariants. The fractional characteristic equations for V_3 are

$$\frac{\frac{(dx)^\beta}{\Gamma(1+\beta)}}{\frac{t^\alpha}{\Gamma(1+\alpha)}} = \frac{\frac{(dt)^\alpha}{\Gamma(1+\alpha)}}{0} = \frac{du}{1}, \tag{2.28}$$

which give the invariants as $X(x,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$ and $\phi(X) = \frac{x^\beta}{\Gamma(1+\beta)} - \frac{t^\alpha}{\Gamma(1+\alpha)}u$.

Theorem 2.1: Under the group of point transformations $X(x,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$ and $\phi(X) = \frac{x^\beta}{\Gamma(1+\beta)} - \frac{t^\alpha}{\Gamma(1+\alpha)}u$ the FBP equation reduces to an ordinary differential equation $\phi'(X) = 1$, which has the general solution as

$$u(x,t) = \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)}\frac{x^\beta}{t^\alpha} - K\frac{\Gamma(1+\alpha)}{t^\alpha} - 1,$$

where K is an arbitrary constant.
Some other solutions obtained using fractional Lie symmetries are also reported.

Chapter 3: Symmetry analysis of time fractional Potential Burgers' equation[2]

Gazizov (2007) investigated the continuous point transformation groups of some fractional differential equations and proposed some prolongation formulae, where the author assumed the existence of both, the fractional derivative as well as the integer order derivative. In this chapter by means of Lie group method we consider the following time-fractional potential Burgers' equation of the form

$$u_t^{(\alpha)} = Au_{xx} + B(u_x)^2, \quad x \in (0,\infty),\ t > 0,\ 0 < \alpha < 1, \tag{3.1}$$

where A and B are real constant parameters. We assume the Lie symmetries of the form

$$\tilde{x} = x + \epsilon\xi(x,t,u) + o(\epsilon^2)$$
$$\tilde{t} = t + \epsilon\tau(x,t,u) + o(\epsilon^2)$$
$$\tilde{u} = u + \epsilon\eta(x,t,u) + o(\epsilon^2),$$

The associated Lie algebra is the vector field of the form

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}.$$

The fractional second order prolongation is

$$pr^{(2)}V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} + \eta^t\frac{\partial}{\partial u_t} + \eta^x\frac{\partial}{\partial u_x} + \eta^0_\alpha\frac{\partial}{\partial u_t^{(\alpha)}} + \eta^{tt}\frac{\partial}{\partial u_{tt}} + \eta^{xx}\frac{\partial}{\partial u_{xx}} \tag{3.2}$$

For the invariance of eqn.(3.1), we must have

$$pr^{(2)}V([\Delta u])|_{[\Delta u]=0} = 0, \tag{3.3}$$

[2] The work is accepted for publication as "Symmetry analysis of time fractional Potential Burgers' equation," *Mathematical Communications*, 22, pp. 1-11, 2017. (In Press)

where $[\Delta u] = u_t^{(\alpha)} - Au_{xx} - B(u_x)^2$,
or equivalently, if

$$(\eta_\alpha^0 - 2Bu_x\eta^x - A\eta^{xx})|_{[\Delta u]=0} = 0. \tag{3.4}$$

The generalised fractional prolongation vector fields η^x, η_α^0 and η^{xx} are given by

$$\eta^x = \eta_x + u_x\eta_u - (\xi_x + u_x\xi_u)u_x + (\tau_x + u_x\tau_u)u_t \tag{3.5}$$

$$\eta_\alpha^0 = \eta_t^\alpha + (\eta_u - \alpha(\tau_t + u_t\tau_u))u_t^\alpha - u(\eta_u)_t^\alpha - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x) + \sum_{n=1}^{\infty}[\binom{\alpha}{n}\partial_t^n\eta_u - \binom{\alpha}{n+1}D_t^{n+1}(\tau)]D_t^{\alpha-n}(u) + \mu,$$

where $\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}(-u)^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}$, and

$$\eta^{xx} = \eta_{xx} + u_x\eta_{ux} - (\xi_{xx} + u_x\xi_{ux})u_x + (\tau_{xx} + u_x\tau_{ux})u_t - 2(\tau_x + u_x\tau_u)u_{xt} + [\eta_{xu} + u_x\eta_{uu} - (\xi_{xu} + u_x\xi_{uu})u_x - (\tau_{xu} + u_x\tau_{uu})u_t]u_x + u_{xx}(\eta_u - u_x\xi_u - u_t\tau_u) - 2(\xi_x + u_x\xi_u)u_{xx}$$

Now, using the above generalised prolongation vector fields in eqn. (3.4) and equating the coefficient of various derivative terms to zero, we get the following simplified set of determining equations

$$\tau_u = 0 \tag{3.6}$$

$$\tau_x = 0 \tag{3.7}$$

$$\xi_u = 0 \tag{3.8}$$

$$\binom{\alpha}{n}\partial_t^n\eta - \binom{\alpha}{n+1}D_t^{n+1}(\tau) = 0, \quad n = 1, 2, 3, ... \tag{3.9}$$

$$2\xi_x - \alpha\tau_u = 0 \tag{3.10}$$

$$2B\xi_x - B\alpha\tau_u - B\tau_u = 0 \tag{3.11}$$

$$D_t^n(\xi) = 0, \quad n = 1, 2, 3, ... \tag{3.12}$$

On solving the above eqns., we obtain the infinitesimals as

$$\xi = c_1x + c_2 \tag{3.13}$$

$$\tau = \frac{2c_1t}{\alpha} \tag{3.14}$$

$$\eta = c_3 \tag{3.15}$$

where c_1,c_2,c_3 are arbitrary constant parameters. The point symmetry generators admitted by the eqn. (3.1) are given by

$$V_1 = x\frac{\partial}{\partial x} + \frac{2t}{\alpha}\frac{\partial}{\partial t} \tag{3.16}$$

$$V_2 = \frac{\partial}{\partial x} \tag{3.17}$$

$$V_3 = \frac{\partial}{\partial u} \tag{3.18}$$

Hence, the infinitesimal operator becomes

$$V = (c_1 x + c_2)\frac{\partial}{\partial x} + \frac{2t}{\alpha}\frac{\partial}{\partial t} + c_3\frac{\partial}{\partial u}.$$

It is easy to check that the vector fields $\{V_1, V_2, V_3\}$ are closed under the Lie bracket. Thus a basis for Lie algebra is $\{V_1, V_2, V_3\}$, which consists of three two dimensional subalgebra $\{V_1, V_2\}$,$\{V_2, V_3\}$ and $\{V_3, V_1\}$.
The corresponding one parameter groups are

$$g_1 : (x, t, u) \to (e^{\epsilon}x, e^{\frac{2}{\alpha}}t, u) \tag{3.19}$$

$$g_2 : (x, t, u) \to (x + \epsilon, t, u) \tag{3.20}$$

$$g_3 : (x, t, u) \to (x, t, u + \epsilon) \tag{3.21}$$

Reduction to ODE:

For the infinitesimal generator V_1 the characteristic equations are

$$\frac{dx}{x} = \frac{\alpha dt}{2t} = \frac{du}{0},$$

which give the invariants as $u(x,t) = f(z)$, $z = xt^{\frac{-\alpha}{2}}$. Corresponding to these invariants we can reduce eqn. (3.1) to an ODE of fractional order. We summarize the result in the following theorem :

Theorem 3.1: The similarity transformation $u(x,t) = f(z)$ along with the similarity variable $z = xt^{\frac{-\alpha}{2}}$ reduces the time fractional potential Burgers' equation to the ordinary differential equation of fractional order of the form

$$(P^{1-\alpha,\alpha}_{\frac{2}{\alpha}} f)(z) = A\frac{d^2 f}{dz^2} + B(\frac{df}{dz})^2 \tag{3.22}$$

with the Erdélyi-Kober fractional differential operator

$$(P_\delta^{\tau,\alpha} f)(z) = \prod_{j=0}^{m-1} (\tau + j - \frac{1}{\delta} z \frac{d}{dz})(K_\delta^{\tau+\alpha, m-\alpha} f)(z),\ z > 0,\ \delta > 0,\ \alpha > 0, \tag{3.23}$$

$m = \begin{cases} [\alpha] + 1, & \alpha \notin \mathbb{N} \\ \alpha, & \alpha \in \mathbb{N}. \end{cases}$, where

$$(K_\delta^{\tau,\alpha} f)\ (z) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^\infty (\nu - 1)^{\alpha-1} \nu^{-(\tau+\alpha)} f(z\nu^{\frac{1}{\delta}})\, d\nu\,, & \alpha > 0; \\ f(z), & \alpha = 0 \end{cases} \tag{3.24}$$

is the Erdélyi-Kober fractional integral operator [34].
As the order $0 < \alpha < 1$ of the above equation is arbitrary, there exists no method to solve the above differential equation of fractional order in general. However, when $B = 0$,the group invariant solution of eqn. (3.1) has the form
$u(x,t) = K_1 W(-\frac{xt^{-\frac{\alpha}{2}}}{\sqrt{A}}; -\frac{\alpha}{2}, 1) + K_2 W(\frac{xt^{-\frac{\alpha}{2}}}{\sqrt{A}}; -\frac{\alpha}{2}, 1)$, where K_1 and K_2 are arbitrary parameters and $W(z; \lambda, \mu)$ is the Wright function given by $W(z; \lambda, \mu) = \sum_{i=0}^{\infty} \frac{z^i}{i!\Gamma(\lambda i + \mu)}$.

We also utilised the invariant subspace method to furnish an exact solution of time-fractional potential Burgers' equation as

$$u(x,t) = a_1(t) + a_2(t)x + a_3(t)x^2, \tag{3.25}$$

where $a_1(t)$,$a_2(t)$ and $a_3(t)$ are given by

$$-\frac{1}{a_3(t)} = \frac{2}{\Gamma(1+\alpha)} \int B dt^\alpha + s_1, \tag{3.26}$$

$$\log a_2(t) = \frac{4}{\Gamma(1+\alpha)} \int a_3(t) B dt^\alpha + s_2 \tag{3.27}$$

and

$$a_1(t) = \frac{1}{\Gamma(1+\alpha)} [\int (2a_3(t)A + (a_2(t))^2 B) dt^\alpha] + s_3, \tag{3.28}$$

where s_1,s_2 and s_3 are arbitrary constants.

Chapter 4: Similarity reduction and exact solutions of a variable coefficient space-time fractional potential Burgers' equation[3]

Herein, we investigate the symmetries and reductions of space-time fractional Potential Burgers' equation

$$u_t^{(\alpha)} = f(t) u_x^{(2\beta)} + g(t) (u_x^{(\beta)})^2, \quad x \in (0, \infty),\ t > 0,\ 0 < \alpha, \beta < 1, \tag{4.1}$$

[3] The work is accepted for publication as "Symmetry classification and exact solutions of a variable coefficient space-time fractional Potential Burgers' equation," *International Journal of Differential Equations*, 2016. (In Press)

where $f(t)$ and $g(t)$ are arbitrary functions of t. Here the invariance condition provides the set of determining equations as

$$\tau_u = 0 \tag{4.2}$$

$$\tau_x^{(\beta)} = 0 \tag{4.3}$$

$$\xi_u = 0 \tag{4.4}$$

$$2g(t)\xi_x^{(\beta)} - g(t)\tau_t^{(\alpha)} - \tau g_t^{(\alpha)} - g(t)\eta_u - f(t)\eta_{uu} = 0 \tag{4.5}$$

$$2f(t)\xi_x^{(\beta)} - f(t)\tau_t^{(\alpha)} - \tau f_t^{(\alpha)} = 0 \tag{4.6}$$

$$f(t)\xi_x^{(2\beta)} - \xi_t^{(\alpha)} - 2g(t)\eta_x^{(\beta)} - 2f(t)(\eta_u)_x^{(\beta)} = 0 \tag{4.7}$$

$$\eta_t^{(\alpha)} - f(t)\eta_x^{(2\beta)} = 0 \tag{4.8}$$

On solving eqn. (4.8) by using fractional Lie group method we obtain a particular solution as

$$\eta = a_1\frac{x^\beta}{\Gamma(1+\beta)} + a_2F_t^{(\alpha)}(t) + a_2\frac{x^{2\beta}}{\Gamma(1+2\beta)} + a_3$$

where $F_t^{(2\alpha)} = f_t^{(\alpha)}$. Using this value of η in eqns. (4.5) and (4.6), we get

$$2(g(t) - f(t))\xi_x^{(2\beta)} = 0$$

which brings forth the following possibilities:

(i)$\xi_x^{(2\beta)} = 0$ and (ii) $g(t) = f(t)$

Case (i): In this case, from the determining equations, we get

$$\xi = -2a_1G(t) - 2a_2G(t)\frac{x^\beta}{\Gamma(1+\beta)} + a_4\frac{x^\beta}{\Gamma(1+\beta)} + a_5, \tag{4.9}$$

$$\tau = \frac{1}{F_t^{(2\alpha)}(t)} - [4a_2H(t)_+2a_4F_t^{(\alpha)}(t) + a_6], \tag{4.10}$$

$$\eta = a_1\frac{x^\beta}{\Gamma(1+\beta)} + a_2F_t^{(\alpha)}(t) + a_2\frac{x^{2\beta}}{\Gamma(1+2\beta)} + a_3, \tag{4.11}$$

where $G_t^{(\alpha)}(t) = g(t)$, $H_t^{(\alpha)}(t) = F_t^{(2\alpha)}(t)G(t)$, and $a_1, a_2, ..., a_6$ are six arbitrary constants. Using eqns. (4.9-4.11) in eqn. (4.5) we also get $g(t) = kf(t)$, where k is an arbitrary constant. This covers the case (ii), as for $f(t) = g(t)$ on solving the determining equations, we get $\xi_x^{(2\beta)} = 0$. Further for $f(t) = g(t) = 1$ and $\beta = 1$ the infinitesimals can be reduced to those reported by Wu, by setting the coefficients $a_1 = -c_5$, $a_2 = -2c_6$, $a_3 = c_3$, $a_4 = c_4$, $a_5 = c_1$, $a_6 = c_2$.

Hence, the fractional point symmetry generators admitted by the eqn. (4.1) are given by

$$V_1 = -2G(t)\frac{\partial^\beta}{\partial x^\beta} + \frac{x^\beta}{\Gamma(1+\beta)}\frac{\partial}{\partial u}$$

$$V_2 = -2G(t)\frac{x^\beta}{\Gamma(1+\beta)}\frac{\partial^\beta}{\partial x^\beta} - \frac{4H(t)}{F_t^{(2\alpha)}(t)}\frac{\partial^\alpha}{\partial t^\alpha} + [F_t^{(\alpha)}(t) + \frac{x^{2\beta}}{\Gamma(1+2\beta)}\frac{\partial}{\partial u}]$$

$$V_3 = \frac{\partial}{\partial u}$$

$$V_4 = \frac{x^\beta}{\Gamma(1+\beta)}\frac{\partial^\beta}{\partial x^\beta} + \frac{2F_t^{(\alpha)}(t)}{F_t^{(2\alpha)}(t)}\frac{\partial^\alpha}{\partial t^\alpha}$$

$$V_5 = \frac{\partial^\beta}{\partial x^\beta},$$

and

$$V_6 = \frac{1}{F_t^{(2\alpha)}(t)}\frac{\partial^\alpha}{\partial t^\alpha}$$

These infinitesimal generators form a six-dimensional fractional Lie algebra under the Lie bracket $[X, Y] = XY - YX$.The commutator table is as given below:

	V_1	V_2	V_3	V_4	V_5	V_6
V_1	0	0	0	V_1	-V_3	$2V_5$
V_2	0	0	0	$2V_2$	$2V_1$	$4V_4$-$2V_3$
V_3	0	0	0	0	0	0
V_4	-V_1	-$2V_2$	0	0	-V_5	$2V_6$
V_5	V_3	-$2V_1$	0	V_5	0	0
V_6	-$2V_5$	$2V_3$-$4V_4$	0	-$2V_6$	0	0

Further, from the commutator table it can be seen that the sets $\{V_3\}$ and $\{V_1, V_2, V_3\}$ form solvable subalgebras. Also, V_3 is the centre of the six-dimensional Lie algebra as it commutes with every element of the Lie algebra. These are the solutions obtained using fractional Lie symmetries corresponding to various infinitesimal generators

$$u(x,t) = \frac{1}{2G(t)}\left[e^{\int H_1 dT}\left(k_1 - \int H_2 e^{-\int H_1 dT} dT\right) - \frac{x^{2\beta}}{\Gamma(1+2\beta)}\right] \tag{4.12}$$

$$u(x,t) = \frac{1}{k}\log\left\{k_2 + 2\sqrt{\pi}k\,\mathrm{erf}\left(\frac{x^\beta}{2\Gamma(1+\beta)\sqrt{F_t^{(\alpha)}}}\right)\right\} + k_3. \tag{4.13}$$

$$u(x,t) = \frac{m}{n}\left[\frac{x^\beta}{\Gamma(1+\beta)} + \frac{m}{n}G(t) - k_4\right]. \tag{4.14}$$

$$u(x,t) = \frac{1}{k}\log\left\{ e^{\frac{r^2}{\Gamma(1+\alpha)}\left\{\frac{x^\beta}{r\Gamma(1+\beta)} - \frac{F_t^{(\alpha)}}{\Gamma(1+\alpha)}\right\}} - ke^{\frac{c_1 r^2}{\Gamma(1+\alpha)}} \right\} - \frac{r^2}{\Gamma(1+\alpha)}\left\{\frac{x^\beta}{r\Gamma(1+\beta)} - \frac{F_t^{(\alpha)}}{\Gamma(1+\alpha)}\right\} + c_2 \quad (4.15)$$

$$u(x,t) = \left[F_t^{(\alpha)} + \frac{1}{k}\log\left\{\cosh\sqrt{k}(c_3 + \frac{x^\beta}{\Gamma(1+\beta)})\right\} - c_4\right], \quad (4.16)$$

where r, s, m and n are non-zero arbitrary constant parameters and k_1, k_2, k_3, k_4, c_1, c_2, c_3, c_4 are arbitrary constants. Some other exact solutions of eqn. (4.1) are also addressed in this chapter.

Chapter 5: Lie group of transformations of time fractional Gardner equation

In this work, we investigated the Lie symmetries of the time-fractional Gardner equation of the form

$$u_t^{(\alpha)} = Auu_x + Bu^2u_x + u_{xxx}, \quad x \in (0,\infty),\ t > 0,\ 0 < \alpha < 1, \quad (5.1)$$

where A and B are real constant parameters.
On solving the set of determining equations obtained from the invariance condition for time-fractional Gardner equation, we arrive at the following two cases (i) $A = 0$ or (ii) $B = 0$
Case (i) if $A = 0$

$$\xi = c_1x + c_2 \quad (5.2)$$

$$\tau = \frac{3c_1t}{\alpha} \quad (5.3)$$

$$\eta = -c_1u \quad (5.4)$$

where c_1, c_2 are arbitrary parameters. The point symmetry generators admitted by the time-fractional Gardner equation are given by

$$V_1 = x\frac{\partial}{\partial x} + \frac{3t}{\alpha}\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} \quad (5.5)$$

$$V_2 = \frac{\partial}{\partial x} \quad (5.6)$$

Case (ii) if $B = 0$

$$\xi = c_1x + c_2 \quad (5.7)$$

$$\tau = \frac{3c_1 t}{\alpha} \tag{5.8}$$

$$\eta = -2c_1 u \tag{5.9}$$

where c_1, c_2 are arbitrary parameters. And the point symmetry generators are

$$V_1 = x\frac{\partial}{\partial x} + \frac{3t}{\alpha}\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}, \tag{5.10}$$

$$V_2 = \frac{\partial}{\partial x}. \tag{5.11}$$

Reduction to ODE:

For the infinitesimal generator

$$V_1 = x\frac{\partial}{\partial x} + \frac{3t}{\alpha}\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} \tag{5.12}$$

the characteristic equations are

$$\frac{dx}{x} = \frac{\alpha dt}{3t} = \frac{du}{-u},$$

which give the invariants as $u(x,t) = x^{-1}f(z)$, $z = xt^{\frac{-\alpha}{3}}$.
Similarly for the infinitesimal generator

$$V_1 = x\frac{\partial}{\partial x} + \frac{3t}{\alpha}\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} \tag{5.13}$$

the invariants are $u(x,t) = x^{-2}f(z)$, $z = xt^{\frac{-\alpha}{3}}$ Corresponding to these invariants we can reduce eqn. (5.1) to an ODE of fractional order. We summarize the result in the following two theorems :

Theorem 5.1: The similarity transformation $u(x,t) = x^{-1}f(z)$ along with the similarity variable $z = xt^{\frac{-\alpha}{3}}$ reduces the time fractional Gardner equation to the ordinary differential equation of fractional order of the form

$$(P^{1-\frac{5}{3}\alpha,\alpha}_{\frac{3}{\alpha}} f)(z) = \frac{d^3 f}{dz^3} + Bf^2\frac{df}{dz} \tag{5.14}$$

with the Erdélyi-Kober fractional differential operator

$$(P^{\tau,\alpha}_{\delta} f)(z) = \prod_{j=0}^{m-1}(\tau + j - \frac{1}{\delta}z\frac{d}{dz})(K^{\tau+\alpha,m-\alpha}_{\delta} f)(z),\ z > 0,\ \delta > 0,\ \alpha > 0, \tag{5.15}$$

$$m = \begin{cases} [\alpha]+1, & \alpha \notin \mathbb{N} \\ \alpha, & \alpha \in \mathbb{N}. \end{cases}, \quad \text{where}$$

$$(K_\delta^{\tau,\alpha} f)\,(z) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^\infty (\nu-1)^{\alpha-1} \nu^{-(\tau+\alpha)} f(z\nu^{\frac{1}{\delta}})\, d\nu\,, & \alpha > 0; \\ f(z), & \alpha = 0 \end{cases} \tag{5.16}$$

is the Erdélyi-Kober fractional integral operator.

Theorem 5.2: The similarity transformation $u(x,t) = x^{-2} f(z)$ along with the similarity variable $z = xt^{\frac{-\alpha}{3}}$ reduces the time fractional Gardner equation to the ordinary differential equation of fractional order of the form

$$(P_{\frac{3}{\alpha}}^{1-\frac{4}{3}\alpha,\alpha} f)(z) = \frac{d^3 f}{dz^3} + Af\frac{df}{dz} \tag{5.17}$$

with the Erdélyi-Kober fractional differential operator

$$(P_\delta^{\tau,\alpha} f)(z) = \prod_{j=0}^{m-1} (\tau + j - \frac{1}{\delta} z \frac{d}{dz})(K_\delta^{\tau+\alpha, m-\alpha} f)(z),\ z > 0,\ \delta > 0,\ \alpha > 0, \tag{5.18}$$

$$m = \begin{cases} [\alpha]+1, & \alpha \notin \mathbb{N} \\ \alpha, & \alpha \in \mathbb{N}. \end{cases}, \quad \text{where}$$

$$(K_\delta^{\tau,\alpha} f)\,(z) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^\infty (\nu-1)^{\alpha-1} \nu^{-(\tau+\alpha)} f(z\nu^{\frac{1}{\delta}})\, d\nu\,, & \alpha > 0; \\ f(z), & \alpha = 0 \end{cases} \tag{5.19}$$

is the Erdélyi-Kober fractional integral operator.
Further, some exact solutions of eqn. (5.1) have also been reported in this chapter.

Chapter 6: Group classification of space-time fractional coupled KdV equation

Here, we study the coupled KdV equation

$$u_t^{(\alpha)} + f(t) u u_x^{(\beta)} + g(t) v v_x^{(\beta)} + h(t) u_x^{(3\beta)} = 0, \tag{6.1}$$

$$v_t^{(\alpha)} + \delta(t) u v_x^{(\beta)} + k(t) v_x^{(3\beta)} = 0, \tag{6.2}$$

where $x \in (0,\infty),\ t > 0,\ 0 < \alpha, \beta < 1$.
We assume the Lie symmetries of the form

$$\frac{\tilde{x^\beta}}{\Gamma(1+\beta)} = \frac{x^\beta}{\Gamma(1+\beta)} + \epsilon \xi(x,t,u,v) + o(\epsilon^2) \tag{6.3}$$

$$\frac{\tilde{t^{\alpha}}}{\Gamma(1+\alpha)} = \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \epsilon\tau(x,t,u,v) + o(\epsilon^2) \tag{6.4}$$

$$\tilde{u} = u + \epsilon\eta^1(x,t,u,v) + o(\epsilon^2) \tag{6.5}$$

$$\tilde{v} = v + \epsilon\eta^2(x,t,u,v) + o(\epsilon^2) \tag{6.6}$$

where ϵ is the group parameter and ξ, τ and η^1, η^2 are the infinitesimals of the transformations for the independent and dependent variables, respectively. The associated Lie algebra is

$$V = \xi(x,t,u,v)\frac{\partial^{\beta}}{\partial x^{\beta}} + \tau(x,t,u,v)\frac{\partial^{\alpha}}{\partial t^{\alpha}} + \eta^1(x,t,u,v)\frac{\partial}{\partial u} + \eta^2(x,t,u,v)\frac{\partial}{\partial v}. \tag{6.7}$$

The infinitesimals for the coupled KdV equation are obtained as

$$\xi = \frac{1}{H_t^{(\alpha)}}[3k_1H + k_4] \tag{6.8}$$

$$\tau = k_1\frac{x^{\beta}}{\Gamma(1+\beta)} + k_5, \tag{6.9}$$

$$\eta^1 = k_2u, \tag{6.10}$$

$$\eta^2 = k_3v, \tag{6.11}$$

where $k_1, ..., k_5$ are arbitrary constants and $k(t) = h(t) = H_t^{(\alpha)}$, with

$$(\xi f)_t^{(\alpha)} - (k_1 + k_2)f = 0, \tag{6.12}$$

$$(\xi g)_t^{(\alpha)} - (2k_3 + k_1 - k_2)g = 0, \tag{6.13}$$

$$(\xi \delta)_t^{(\alpha)} - (k_1 + k_2)\delta = 0 \tag{6.14}$$

Hence, the fractional point symmetry generators admitted by the coupled KdV equation are given by

$$V_1 = \frac{3H}{H_t^{(\alpha)}}\frac{\partial^{\alpha}}{\partial t^{\alpha}} + \frac{x^{\beta}}{\Gamma(1+\beta)}\frac{\partial^{\beta}}{\partial x^{\beta}}$$

$$V_2 = u\frac{\partial}{\partial u}$$

$$V_3 = v\frac{\partial}{\partial v}$$

$$V_4 = \frac{1}{H_t^{(\alpha)}}\frac{\partial^{\alpha}}{\partial t^{\alpha}}$$

$$V_5 = \frac{\partial^{\beta}}{\partial x^{\beta}}.$$

Further the set $\{V_1, V_2, V_3, V_4, V_5\}$forms a five dimensional Lie algebra under the Lie bracket $[X, Y] = XY - YX$.

Some exact solutions of the fractional coupled KdV equation

We also investigated some exact solutions of the fractional coupled KdV equation corresponding to the following infinitesimal generators
(i)$V_1 + aV_2 + bV_3$
(ii)$V_2 + mV_3 + nV_4$,
where a, b, m and n are arbitrary constant parameters.

(i) For the infinitesimal generator

$$V_1 + aV_2 + bV_3 = \frac{3H}{H_t^{(\alpha)}}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{x^\beta}{\Gamma(1+\beta)}\frac{\partial^\beta}{\partial x^\beta} + au\frac{\partial}{\partial u} + bv\frac{\partial}{\partial v}$$

we reduced the fractional coupled KdV equation to the following coupled nonlinear equations

$$F'''(X) - c_1F(X)F'(X) + c_2G(X)G'(X) - \frac{\zeta F'(X)}{3} - \frac{aF(X)}{3} = 0 \tag{6.15}$$

$$G'''(X) + c_3F(X)G'(X) - \frac{\zeta G'(X)}{3} - \frac{bG(X)}{3} = 0 \tag{6.16}$$

To solve the reduced system, we seek a special solution of the form

$$F(X) = A_0 + A_1X + A_2X^2 + A_3\frac{1}{X} + A_4\frac{1}{X^2},$$

$$G(X) = B_0 + B_1X + B_2X^2 + B_3\frac{1}{X} + B_4\frac{1}{X^2},$$

where $A_i, B_i,\ i = 0, 1, ..., 4$ are arbitrary constants.
Substituting these expressions for F and G in the reduced system we get the following exact solution to the coupled eqns. (6.1) and (6.2)

$$u(x,t) = \frac{\frac{x^{3\beta}}{(\Gamma(1+\beta))^3} - 24H(t)}{2c_3\frac{x^{2\beta}}{(\Gamma(1+\beta))^2}\sqrt{H(t)}} \tag{6.17}$$

$$v(x,t) = \frac{\frac{x^{3\beta}}{(\Gamma(1+\beta))^3} - 24H(t)}{2c_3\frac{x^{2\beta}}{(\Gamma(1+\beta))^2}}\sqrt{\frac{c_3 - c_1}{2H(t)}} \tag{6.18}$$

(ii) For the infinitesimal generator

$$V_2 + mV_3 + nV_4 = u\frac{\partial}{\partial u} + mv\frac{\partial}{\partial v} + \frac{n}{H_t^{(\alpha)}}\frac{\partial^\alpha}{\partial t^\alpha}$$

the reduced form of fractional coupled KdV equation is

$$P'''(X) + c_4P(X)P'(X) + c_5Q(X)Q'(X) - \frac{P(X)}{n} = 0 \tag{6.19}$$

$$Q'''(X) + c_6 P(X) Q'(X) - \frac{mG(X)}{n} = 0 \tag{6.20}$$

which gives the solution

$$u(x,t) = (A_0 + \frac{m}{c_6 n} \frac{x^\beta}{\Gamma(1+\beta)}) e^{-\frac{H(t)}{n}} \tag{6.21}$$

$$v(x,t) = \frac{\sqrt{\frac{m(c_6 - c_4 m)}{c_5}}}{c_6 m n} (A_0 + \frac{m}{c_6 n} \frac{x^\beta}{\Gamma(1+\beta)}) e^{-\frac{mH(t)}{n}} \tag{6.22}$$

Some other linear combinations of infinitesimal generators have also been considered to investigate few more exact solutions. Further the one parameter Lie group of point transformations corresponding to each infinitesimal generator are also reported so that one can easily generate some more solutions of the fractional coupled KdV equation.

Chapter 7: Conclusions

The importance of the fractional order systems due to their occurrence in the study of many processes in science and engineering has been the prime reason for the work put up in this thesis. More specifically, the thesis deals with some nonlinear fractional order partial differential equations representing some interesting physical systems viz. space-time fractional Burgers-Poisson equation, time fractional potential Burgers' equation, variable coefficient space-time fractional potential Burgers' equation, time fractional Gardner and space-time fractional coupled KdV equation, from the view point of their underlying Lie point symmetries. The main purpose of Lie symmetry method is to reduce PDEs to ODEs by introducing suitable similarity variables. Here, similarity analysis has been successfully performed on various nonlinear fractional order partial differential equations. To determine the admissible symmetries two methods- one based on non-differentiable functions and the other one based on differentiable functions, have been utilized. It has been illustrated that the fractional partial differential equations possess similarity solutions, exactly as its counterparts with integer-order derivatives for the first approach, while in the second approach the fractional differential equations possess fewer dimensional Lie algebra than the integer one. In both cases by using conveniently defined similarity variables the fractional equations reduce to ordinary differential equations which are further solved for some group invariant solutions. Also, the Lie group method has been applied on a coupled system of fractional differential equations. Some exact solutions have been deduced using the invariant subspace method.

Finally, it is worth mentioning that in spite of the focus on the exact solutions, the

author found it really difficult at times to handle the reduced system of ODEs for extracting the exact solutions. In some cases the solutions obtained are of very specific nature and further application of Lie group method on the reduced system lead only to trivial symmetries.

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Publications Based on Present Work

Journal Publications:

1. M. Gaur, and K. Singh. "On group invariant solutions of fractional order Burgers-Poisson equation," *Applied Mathematics and Computation*, 244.1, pp. 870-877, 2014.
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2. M. Gaur, and K. Singh. "Symmetry analysis of time fractional Potential Burgers' equation," *Mathematical Communications*, 22, pp. 1-11, 2017. (In Press)
   [SCIE, MathSci Net,Scopus]

3. M. Gaur, and K. Singh. "Symmetry classification and exact solutions of a variable coefficient space-time fractional Potential Burgers' equation," *International Journal of Differential Equations*, 2016. (In Press)
   [MathSci Net, Scopus]

4. M. Gaur, and K. Singh. "Lie group of transformations of time fractional Gardner equation,"(Under Communication).

5. M. Gaur, and K. Singh. "Group classification of space-time fractional coupled KdV equation,"(Under Communication).